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ON A FULLY NONLINEAR PARABOLIC EQUATION AND THE ASYMPTOTIC BEHAVIOUR OF ITS SOLUTIONS

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ABSTRACT

This paper deals with the first boundary problem associated with the fully nonlinear equation $u_t = \mathrm{Min}\{\psi, \Delta u\}$ on the set $\Omega \times (0,\infty)$, where Ω is a domain of $\mathbb{R}^N$ and $\psi(x)$ is a given obstacle such that $\psi \geq 0$ on Ω. Formulating the problem (occurring in heat control) as an Evolution Variational Inequality, H. Brezis obtained the existence and uniqueness of weak solutions in the space $H_0^1(\Omega)$ as well as weak convergence to an unknown equilibrium point of the equation (when t goes to infinity). The strong convergence of the solution to the zero equilibrium point is shown here, provided the obstacle is positive and subharmonic. If in addition $\psi(x) \geq \delta > 0$ then the asymptotic behaviour is completely described in the sense that the solution satisfies the linear heat equation $u_t = \Delta u$ on $(T_0,\infty) \times \Omega$, T_0 being a finite time. To do this the results are first presented for strong solutions (that is, those which satisfy the equation a.e.). The fact that under more regularity on the initial datum the weak solution is also a strong one and certain useful comparison principles are proved by using the theory of accretive operators in Banach spaces.

AMS (MOS) Subject Classifications: 35K55, 49A29, 47H06

Key Words: Nonlinear heat equation, evolution variational inequality, accretive operator, asymptotic behaviour

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SIGNIFICANCE AND EXPLANATION

Problems arising in heat control theory are often modeled by Parabolic Variational Inequalities (PVI) (see G. Duvaut-J. L. Lions [16]). One example of such problems, considered in this paper, corresponds to the case where the temporal temperature variation of a body or fluid Ω of $\mathbb{R}^N$ is not allowed to be greater than a given positive function (called "obstacle").

In an earlier work [8], H. Brezis has proved that the PVI arising in such a situation can be formulated as an abstract Cauchy problem on the space $H_0^1(\Omega)$, and he obtained the existence and uniqueness of solutions by means of the theory of maximal monotone operators. Using this theory, he also proved in [8] that the solution converges weakly in $H_0^1(\Omega)$ to an equilibrium point when t goes to infinity. Similar to other nonlinear evolution equations, there exists a large set of such equilibrium points. An important question is to decide how the solution selects an equilibrium point among all them and whether the convergence to it also holds in the strong topology.

In this paper some answers to both questions are given by setting the problem in a different framework. It is easy to see that solutions being more regular ("strong solutions") satisfy a fully nonlinear parabolic equation. Such strong solutions are obtained via a "dual" problem that is shown to be "well possed" in $L^1(\Omega)$ in the sense that the accretive operators theory can be applied, assuming that the obstacle is sufficiently smooth. It is also shown that the "direct problem" is well possed in $L^\infty(\Omega)$ for more regular obstacles.

Adapting a curious comparison result of Ph. Benilan and J. I Diaz ([3]) some estimates are obtained. Finally it is shown that the solution converges strongly in $H_0^1(\Omega)$ to the zero equilibrium point when the obstacle is assumed to be a subharmonic function on Ω. If in addition the obstacle is strictly positive, then the asymptotic behaviour is completely described because it is shown that the solution verifies the linear heat equation after a sufficiently large time T_0. Different results on the strong convergence and the selection of the equilibrium point are also given.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON A FULLY NONLINEAR PARABOLIC EQUATION AND
THE ASYMPTOTIC BEHAVIOUR OF ITS SOLUTIONS

J. Ildefonso Diaz*

§1. INTRODUCTION

Let Ω be a smooth bounded domain in $\mathbb{R}^N$. Given $\psi \in L^2(\Omega)$ with $\psi \geq 0$ a.e. and $u_0 \in H_0^1(\Omega)$ we consider the problem of finding a function $u(t,x)$ satisfying

$$(1) \qquad \begin{cases} u_t = \mathrm{Min}\{\psi, \Delta u\} & \text{on } (0,\infty) \times \Omega \\ u = 0 & \text{on } (0,\infty) \times \partial\Omega \\ u(0,x) = u_0(x) & \text{on } \Omega \end{cases}$$

Such type of problems occur in heat control (see [16], Chap. 2). Formulations as (1) also appear in a non-standard statement of the Stefan problem (see later Remark A.1) as well as in some particular case of the so called Bellman's equation of Dynamic Programming (see Remark 5).

Problem (1) can be expressed in a weak form by means of the following Evolution Variational Inequality

$$(2) \qquad \begin{aligned} & u_t \in K, \quad K = \{v \in H_0^1(\Omega) : v \leq \psi \text{ a.e. on } \Omega\} \\ & \int_\Omega u_t(v - u_t)dx + \int_\Omega \mathrm{grad}\, u \cdot \mathrm{grad}(v - u_t)dx \geq 0 \quad \forall\, v \in K \text{ and } t > 0 . \end{aligned}$$

The existence and uniqueness of a solution of (2), for each $u_0 \in H_0^1(\Omega)$, was proved by H. Brezis in [8] (see also [5]). Also the asymptotic behaviour is considered in [8] by means of the abstract result on asymptotic behaviour of solutions of evolutions equations. It is shown there that $u(t,x)$ converges weakly in $H_0^1(\Omega)$, when $t \to \infty$, to a function

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$u_\infty(x) \in H_0^1(\Omega)$ satisfying

(3) $$\operatorname{Min}\{\Delta u_\infty, \psi\} = 0 \quad \text{on} \quad \Omega$$

in the sense that

(4) $$\int_\Omega \operatorname{grad} u_\infty \operatorname{grad} v \, dx \geq 0 \qquad \forall \, v \in K \, .$$

Nevertheless it is neither known how the solution selects an equilibrium point among all of them nor if the convergence also holds in the strong topology of $H_0^1(\Omega)$. Both questions were proposed in [8] and they are, essentially, the main aims of this work.

Our methods for the study of the asymptotic behaviour are based on considerations made in terms of strong solutions i.e. solutions which satisfy (1) a.e. Because of this we will first consider some regularity results. On this respect it is not difficult to see that if the solution u of (2) is such that $\Delta u(t,\cdot) \in L^1(\Omega)$, for $t > 0$, then u is a strong solution. Nevertheless not every solution of (2) is a strong solution. For instance, when $\psi \equiv 0$ and u_0 is such that $\Delta u_0 \geq 0$ in $D'(\Omega)$ it can be directly verified that $u(t,x) = u_0(x)$ $\forall \, t > 0$ and then u is a strong solution iff $\Delta u_0 \in L^1(\Omega)$. We shall show that if $\psi \in H^1(\Omega)$ with $(-\Delta\psi)^- \in L^2(\Omega)$ and $\Delta u_0 \in L^1(\Omega)$, the solution of (2) is a strong one and satisfies $\Delta u \in C([0,\infty) : L^1(\Omega))$. (A stronger regularity result will also be obtained when $\psi \in C^2(\overline{\Omega})$ and $\Delta u_0 \in L^\infty(\Omega)$).

The main result in our study of asymptotic behaviour of the solutions shows the strong convergence, in $H_0^1(\Omega)$, of the solution to the equilibrium point zero provided $\psi > 0$ and $\Delta\psi \geq 0$ a.e. on Ω. If in addition $\psi(x) \geq \delta > 0$ a.e. $x \in \Omega$ (for some δ) then the asymptotic behaviour is completely described in the sense that we show the solution verifies the linear heat equation $u_t = \Delta u$ on $(T_0,\infty) \times \Omega$ for an adequate finite time T_0. Other answers on the strong convergence and the selection of the equilibrium point are also given.

The essential tool in our treatment of (1) is the consideration of the "dual" (or adjoint) problem

$$
P^* \begin{cases} v_t(t,x) - \Delta\beta(x,v(t,x)) = 0 & \text{on} \quad (0,\infty) \times \Omega \\ \beta(x,v(t,x)) = 0 & \text{on} \quad (0,\infty) \times \partial\Omega \\ v(0,x) = v_0(x) & \text{on} \quad \Omega \end{cases}
$$

where

(5) $$\beta(x,r) = -\mathrm{Min}\{\psi(x), -r\} \quad \text{a.e.} \quad x \in \Omega, \quad \forall\, r \in \mathbb{R}\,.$$

The existence of solutions of P^* in $L^1(\Omega)$ implies the existence of strong solutions of (1) using the relation $v = -\Delta u$.[(1)] The former question, that is the existence of solutions of P^*, has been very much studied recently but, as far as we know, the term $\beta(x,r)$ (a maximal monotone graph of $\mathbb{R}^2$ for a.e. $x \in \Omega$) is always taken in the following two cases: a) $\beta(x,r)$ is independent of x, b) $\beta(x,r)$ is onto a.e. $x \in \Omega$ ([9]). Notice that the $\beta(x,r)$ given in (5) is neither in case a) nor in case b). Anyway, using the theory of Variational Inequalities we shall show that P^* is a "well posed" problem in $L^1(\Omega)$ when $\psi \in H^1(\Omega)$ and $(-\Delta\psi)^- \in L^2(\Omega)$.

The strong solutions of (1) satisfy

$$
P \begin{cases} u_t(t,x) + \beta(x,-\Delta u(t,x)) = 0 & \text{on} \quad (0,\infty) \times \Omega \\ u(t,x) = 0 & \text{on} \quad (0,\infty) \times \partial\Omega \\ u(0,x) = u_0(x) & \text{on} \quad \Omega \end{cases}
$$

with β given by (5). We shall show that P is "well posed" on $L^\infty(\Omega)$ when $\psi \in C^2(\overline{\Omega})$; then it is possible to obtain more regular solutions of (1). (P has previously studied in Benilan-Ha [4] when $\beta(x,r)$ is in case a) or b)).

This paper is planned as follows: In Section 2 the existence of strong solutions of (1) is proved when $\psi \in H^1(\Omega)$ and $\Delta\psi$ is a measure such that $(-\Delta\psi)^- \in L^2(\Omega)$; besides, such solutions are shown to be more regular if $\psi \in C^2(\overline{\Omega})$. The arguments of duality

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[(1)] Duality arguments have already been used in G. Diaz-J. I. Diaz [14].

between (1) and P^* are also presented. In Section 3 we show several comparison results of different nature. Finally, in Section 4, the asymptotic behaviour is considered.

Some results of the theory of evolution equations governed by accretive operators in Banach spaces are used through the paper. In several appendices we present a summary of the abstract theory as well as the proofs of the fact that the abstract hypotheses are satisfied when problems P and P^* are studied as abstract Cauchy problems on $L^\infty(\Omega)$ and $L^1(\Omega)$ (or $H^{-1}(\Omega)$) respectively.

§2. ABOUT THE REGULARITY AND THE DUAL PROBLEM.

In the following it is useful to recall the essential part of the proof of the existence and uniqueness of solutions of (2) given in [8]. It is based in the fact that (2) can be equivalently formulated as

$$\text{(6)} \qquad \int_\Omega \operatorname{grad} u \cdot \operatorname{grad}(v - u_t)dx + \varphi(v) - \varphi(u_t) \geq 0 \qquad \forall\, v \in H_0^1(\Omega) \text{ and } t > 0$$

where φ is a convex l.s.c. function defined on $H_0^1(\Omega)$ by

$$\text{(7)} \qquad \varphi(v) = \begin{cases} \frac{1}{2}\int_\Omega |v|^2 dx & \text{if } v \in K \\ +\infty & \text{if } v \in K\,. \end{cases}$$

Introducing the conjugate convex function of φ by

$$\text{(8)} \qquad \varphi^*(x) = \sup_{y \in H_0^1(\Omega)} \left\{ \int_\Omega \operatorname{grad} x \cdot \operatorname{grad} y - \varphi(y) \right\}$$

inequality (6) can be written as $-u \in \partial\varphi(u_t)$ or equivalently

$$\text{(9)} \qquad u_t - \partial\varphi^*(-u) \ni 0\,.$$

By the theory of maximal monotone operators on Hilbert spaces ([7]) it is known that for any $u_0 \in H_0^1(\Omega)$, $u_0 \in \overline{D(-\partial\varphi^*(-\cdot))}^{H_0^1(\Omega)}$ there exists a unique solution $u \in C([0,\infty) : H_0^1(\Omega)) \cap W_{loc}^{1,1}((0,\infty) : H_0^1(\Omega))$ of (9). In addition

(10) $$u(t) \in D(-\partial\varphi^*(-\cdot)) \quad \text{for any} \quad t > 0\,.^{(2)}$$

Finally by the results of [5] (Proposition II.10 and Lemma II.7) we have that $D(-\partial\varphi^*(-\cdot))$ is a dense set in $H_0^1(\Omega)$. So the result of [8] follows.

As it has been pointed out in the Introduction we are interested on the solutions of (1) that satisfy it a.e. Such functions will be termed strong solutions of (1) in contrast to the solutions of (2) or weak solutions. The following lemma enlightens the connection between weak and strong solutions.

Lemma 1. Let $\psi \in L^2(\Omega)$ with $\psi \geq 0$ a.e. and let $u \in C([0,\infty) : H_0^1(\Omega)) \cap W^{1,1}_{loc}(0,\infty : H_0^1(\Omega))$ be such that $\Delta u(t) \in L^1(\Omega)$ a.e. $t > 0$. Then, u is a weak solution of (1) iff u is also a strong solution.

Proof. Suppose u is a weak solution of (1) such that $\Delta u(t) \in L^1(\Omega)$ a.e. $t > 0$. Taking $v = u_t + \zeta$ in (2) with $\zeta \in D^-(\Omega)$, a simple integration by parts shows that $u_t(t) \leq \Delta u(t)$ a.e. and also that u is a strong solution of (1). On the other hand, if $u \in C([0,\infty) : H_0^1(\Omega)) \cap W^{1,1}_{loc}(0,\infty : H_0^1(\Omega))$ satisfies (1) a.e. it it clear that $u_t \in K$ and also

$$-\Delta u\cdot(v - u_t) \geq -u_t(v - u_t) \quad \text{a.e. on } \Omega \text{ for a.e. } t > 0, \ \forall\, v \in K$$

Then it is enough to apply Lemma 2 of Brezis [6] to $F = -\Delta u$, $w = v - u_t$, $h = g = -u_t\cdot(v - u_t)$ and remark that $(F,w) = \int_\Omega \operatorname{grad} u\cdot\operatorname{grad}(v - u_t)dx$. ∎

A first answer about the regularity of the weak solutions of (1) is the following:

Theorem 1. Assume $\psi \in H^1(\Omega)$ such that $\psi \geq 0$ on Ω and $(-\Delta\psi)^- \in L^2(\Omega)$. Let $u_0 \in H_0^1(\Omega)$ with $\Delta u_0 \in L^1(\Omega)$. Then the weak solution u of (1) satisfies $\Delta u \in C([0,\infty) : L^1(\Omega))$.

As we have said in the Introduction, the proof of Theorem 1 comes essentially considering the problem P^* (when β is given by (5)) formulated as an Abstract Cauchy one on the $L^1(\Omega)$ space

---

(2) We identify $u(t,\cdot)$ with $u(t)$.

$$
(11) \qquad \begin{cases} \dfrac{dv}{dt} + Av \ni 0 & \text{in } L^1(\Omega), \text{ on } (0,\infty) \\ v(0) = v_0 \end{cases}
$$

A being the operator on $L^1(\Omega)$ given by

$$
(12) \qquad \begin{cases} D(A) = \{w \in L^1(\Omega) : \beta(x,w(x)) \in W_0^{1,1}(\Omega) \text{ and } \Delta\beta(x,w(x)) \in L^1(\Omega)\} \\ Aw = -\Delta\beta(\cdot,w(\cdot)) \text{ if } w \in D(A) . \end{cases}
$$

The following result is proved in Appendix 2.

<u>Proposition 1.</u> <u>Assume</u> $\psi \in H^1(\Omega)$ <u>such that</u> $\psi \geq 0$ <u>on</u> Ω <u>and</u> $(-\Delta\psi)^- \in L^2(\Omega)$. <u>Then for every</u> $v_0 \in L^1(\Omega)$ <u>there exists</u> $v \in C([0,\infty) : L^1(\Omega))$ <u>unique</u> $L^1(\Omega)$ <u>semigroup solution of</u> P^*.

A first duality result is given by the next lemma.

<u>Lemma 2.</u> <u>Assume</u> $\psi \in H^1(\Omega)$ <u>such that</u> $\psi \geq 0$ <u>on</u> Ω <u>and</u> $(-\Delta\psi)^- \in L^2(\Omega)$. <u>Let</u> B <u>the operator on the</u> $H_0^1(\Omega)$ <u>space given by</u>

$$
(13) \qquad B(\theta) = -\partial\varphi^*(-\theta) \quad \forall\, \theta \in D(-\partial\varphi^*(-\cdot)) \equiv D(B) .
$$

<u>Consider</u> $b \in D(B)$ <u>such that</u> $-\Delta b \in L^2(\Omega)$(3). <u>Denoting</u> $a = (I + \lambda A)^{-1}(-\Delta b)$ <u>for every</u> $\lambda > 0$, <u>then</u> $a \in L^2(\Omega)$ <u>and</u> $(-\Delta)^{-1}a = (I + \lambda B)^{-1}b$.

<u>Proof.</u> The definition of a implies (for instance when $\lambda = 1$)

$$
(14) \qquad \begin{cases} a(x) - \Delta\beta(x,a(x)) = -\Delta b(x) & \text{on } \Omega \\ \beta(x,a(x)) = 0 & \text{on } \partial\Omega . \end{cases}
$$

As it is seen in Appendix 2 (Lemma A.4.) the previous problem can be formulated as a Stationary Variational Inequality. Then the conclusion $a \in L^2(\Omega)$ comes from the hypotheses on ψ. On the other hand, as $L^2(\Omega) \subset H^{-1}(\Omega)$ then $(-\Delta)^{-1}a = b^* \in H_0^1(\Omega) \cap H^2(\Omega)$ and

$$
(15) \qquad b^*(x) - b(x) = \mathrm{Min}\{\psi(x), \Delta b^*(x)\} .
$$

From (15) $b^* - b \in K$ and besides

(3) For simplicity in the notation we identify $-\Delta$ with the canonical isomorphism Λ from $H_1^0(\Omega)$ onto its dual $H^{-1}(\Omega)$.

$$b^*(x) - b(x) = \Delta b^*(x) \quad \text{a.e.} \quad x \in \{x \in \Omega : (b^*(x) - b(x)) < \psi(x)\}$$

$$b^*(x) - b(x) \leq \Delta b^*(x) \quad \text{a.e.} \quad x \in \{x \in \Omega : (b^*(x) - b(x)) = \psi(x)\}$$

Then, for every $v \in K$ we have

$$\int_\Omega (-\Delta b^* + (b^* - b))(v - (b^* - b))dx \geq 0$$

and integrating by parts

$$\int_\Omega \text{grad } b^* \cdot \text{grad}(v - (b^* - b))dx + \varphi(v) - \varphi(b^* - b) \geq 0$$

namely $-b^* \in \partial\varphi(b^* - b)$ i.e. $b^* + B(b^*) \ni b$. ∎

We are ready now to prove Theorem 1.

<u>Proof of Theorem 1.</u> From Proposition 1 it is enough to show that if u is the weak solution of (1) then $-\Delta u(t)$ coincides with $v(t)$ the unique $L^1(\Omega)$ semigroup solution of P^* corresponding to the initial datum $v_0 = -\Delta u_0$. It is done in two steps: a) $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and b) u_0 in the general case.

<u>Case a).</u> By definition $v(t) = \lim_{n\to\infty} v_n(t)$ where $v_n(t)$ are piecewise constant functions defined by $v_n(t) = a_k^n$ for $k\lambda_n \leq t < (k + 1)\lambda_n$, $a_k^n \in D(A)$ satisfying

$$\frac{a_k^n - a_{k-1}^n}{\lambda_n} + Aa_k^n = 0 \qquad k = 1,\ldots,n$$

$$a_0^n = -\Delta u_0$$

and $\lambda_n > 0$ being such that $\lambda_n < 0$. It is clear that $a_k^n = (I + \lambda_n A)^{-n}(-\Delta u_0)$ and then $a_k^n \in L^2(\Omega)$ because of Lemma 2. Defining $b_k^n = (-\Delta)^{-1}a_k^n$, $b_k^n = (I + \lambda B)^{-n}u_0$ holds. Therefore by defining $u_n(t) = (-\Delta)^{-1}u_n(t) = b_k^n$ for $k\lambda_n \leq t < (k + 1)\lambda_n$ we have

$$-\Delta u(t) = -\Delta(\lim u_n(t)) = \lim(-\Delta u_n(t)) = \lim v_n(t) = v(t)$$

for the m-accretiveness of B in $H_0^1(\Omega)$.

<u>Case b).</u> Let $u_{0,m} \in H_0^1(\Omega) \cap H^2(\Omega)$ be such that $\Delta u_{0,m} \to \Delta u_0$ in $L^1(\Omega)$ as well as in $H^{-1}(\Omega)$ (obviously then $u_{0,m} \to u_0$ in $H_0^1(\Omega)$) when $m \to \infty$. As the semigroup generated

by B is continuous on $H_0^1(\Omega)$ it follows that $u_m(t) \to u(t)$ in $H_0^1(\Omega)$ (then $\Delta u_m(t) \to \Delta u(t)$ in $H^{-1}(\Omega)$), being u, and u_m being the solutions of (1) for the initial datum u_0 and $u_{0,m}$ respectively. Analogously, by the continuity in $L^1(\Omega)$, of the semigroup generated by A we have $-\Delta u_m(t) \to v(t)$ in $L^1(\Omega)$, where $v(t)$ is the solution of P^* with respect to the initial datum $v_0 = -\Delta u_0$. Therefore $v(t) = -\Delta u(t)$ for any $t > 0$. ■

<u>Remark 1.</u> Another regularity result follows by using different methods. Precisely if $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$, $\psi \geq 0$ on Ω, and $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ then the weak solution u of (1) verifies $u \in L^\infty_{loc}(0,\infty ; H^2(\Omega))$ (see Remark II. and Theorem II.13 of Brezis [5]). It is clear that Theorem 1 improves Brezis' result because it can be applied to a wider class of obstacles and initial data (for instance when $\psi(x) \geq \delta > 0$ on Ω for some δ).

Supplementary hypotheses allow us to find a more regular solution of (1).

<u>Theorem 2.</u> <u>Assume</u> $\psi \in C^2(\overline{\Omega})$, $\psi \geq 0$ <u>on</u> $\overline{\Omega}$. <u>Let</u> $u_0 \in H_0^1(\Omega)$ <u>be such that</u> $\Delta u_o \in L^\infty(\Omega)$. <u>Then the weak solution</u> u <u>of (1) satisfies</u>

$$u \in W^{1,\infty}((0,\infty) \times \Omega) \cap L^\infty(0,\infty ; H^2(\Omega)) \ \underline{\text{and}} \ \Delta u(t) \in L^\infty(\Omega) \ \underline{\text{a.e.}} \ t > 0 .$$

To prove Theorem 2 we consider (1) (or equivalently P with β given by (5)) as an Abstract Cauchy Problem on $L^\infty(\Omega)$, i.e.

$$(16) \quad \begin{cases} \dfrac{d\overline{u}}{dt} + C\overline{u} \ni 0 \ \text{ in } \ L^\infty(\Omega), \ \text{ on } \ (0,\infty) \\ \overline{u}(0) = \overline{u}_0 \end{cases}$$

C being the operator on $L^\infty(\Omega)$ given by

$$(17) \quad \begin{aligned} &D(C) = \{w \in L^\infty(\Omega) \cap H_0^1(\Omega) : \Delta w \in L^\infty(\Omega), \ \mathrm{Min}\{\psi,\Delta w\} \in L^\infty(\Omega)\} \\ &Cw = -\mathrm{Min}\{\psi,\Delta w\} \ \text{ if } \ w \in D(C) \end{aligned}$$

The two results stated below are needed for the Proof of Theorem 2 the first one being shown in the Appendix 3.

<u>Proposition 2.</u> <u>Assume</u> $\psi \in C^2(\overline{\Omega})$, $\psi \geq 0$ <u>on</u> $\overline{\Omega}$. <u>Let</u> $\overline{u}_0 \in H_0^1(\Omega)$ <u>be such that</u> $\Delta \overline{u}_0 \in L^\infty(\Omega)$. <u>Then there exists</u> $\overline{u} \in C([0,\infty) ; L^\infty(\Omega))$ <u>unique</u> $L^\infty(\Omega)$- <u>semigroup solution</u>

of (16) (or P). Moreover $\bar{u} \in W^{1,\infty}((0,\infty) \times \Omega) \cap L^\infty(0,\infty; H^2(\Omega))$ *and* $\Delta\bar{u} \in L^\infty(]0,\infty[\times \Omega)$.

Lemma 3. Let $b \in D(B) \cap D(C)$. *Setting* $c = (I + \lambda C)^{-1}b$, *then* $c = (I + \lambda B)^{-1}b$, *for any* $\lambda > 0$.

Proof. From the definition of c it follows that

$$\begin{cases} c(x) - \lambda \mathrm{Min}\{\psi(x), \Delta c(x)\} = b(x) & \text{on } \Omega \\ c(x) = 0 & \text{on } \partial\Omega . \end{cases}$$

As $c - b \in K$ (because $c \in D(C)$) it is easily seen that $-c \in \partial\varphi(c - b)$ proceeding as in Lemma 2. ∎

Proof of Theorem 2. It is enough to see that the weak solution u of (1) coincides with the $L^\infty(\Omega)$ semigroup solution, $\bar{u}$, of (16) corresponding to the initial datum $u_0 = \bar{u}_0$. Without loss of generality we suppose $u_0 \in D(B)$. By definition $\bar{u}(t) = \lim_{n\to\infty} \bar{u}_n(t)$ where $\bar{u}_n(t)$ are piecewise constant functions given by $\bar{u}_n(t) = b_k^n$ for $k\lambda_n \leq t < (k+1)\lambda_n$, $b_k^n \in D(C)$, satisfying

$$\begin{cases} \dfrac{b_k^n - b_{k-1}^n}{\lambda_n} + Cb_k^n = 0 & k = 1,\ldots,n \\ b_0^n = u_0 \end{cases}$$

(or equivalently $b_k^n = (I + \lambda_n C)^{-n}u_0$) when $\lambda_n > 0$ is such that $\lambda_n \to 0$. Thanks to Lemma 3 it is known that $b_k^n = (I + \lambda_n B)^{-n}u_0$. On the other hand, B being m-accretive in $H_0^1(\Omega)$, $u(t) = \lim_{n\to\infty} \bar{u}_n(t)$ in $H_0^1(\Omega)$ and $u(t) = \bar{u}(t)$ holds. ∎

§3. COMPARISON RESULTS.

The following comparison results will be used in the next section under the present formulation which is not the most general one that we could consider.

Let us start with two lemmas.

Lemma 4. *Let* $\psi \in L^2(\Omega)$ *with* $\psi > 0$ *a.e. on* Ω *and let* $u_0 \in H_0^1(\Omega)$. *Set* $h(t,x,\tau,v)$ *be the solution of the heat equation*

$$(18)\qquad \begin{cases} h_t = \Delta h & \text{on } (\tau,\infty) \times \Omega \\ h = 0 & \text{on } (\tau,\infty) \times \partial\Omega \\ h(\tau,x) = v(x) & \text{on } \Omega . \end{cases}$$

Then if u *is the weak solution of (1) we have*

$$(19)\qquad u(t,x) \leq \min\{u_0(x) + t\psi(x), h(t,x,0,u_0)\} \quad \text{a.e. } x \in \Omega \text{ and } t > 0 .$$

Proof. By the regularizing effect (10) we know that for any $t > 0$, $u(t) \in D(B)$ and so $\frac{du}{dt}(t) \in K$ i.e. $\frac{du}{dt}(t) \leq \psi$ a.e. on Ω. Integrating on the t-variable it follows $u(t) - u_0 \leq t\psi(\cdot)$. To show the inequality $u \leq h$ let $\zeta \in L^2_{loc}(0,\infty : H_0^1(\Omega))$ be such that $\zeta(t,\cdot) \geq 0$ a.e. $t > 0$ and $x \in \Omega$. Then $v \equiv \frac{du}{dt} - \zeta \in K$ and substituting in (2) we have

$$\int_\Omega u_t \cdot \zeta dx + \int_\Omega \text{grad } u \cdot \text{grad}(v - u_t) dx \leq 0 .$$

On the other hand

$$\int_\Omega h_t \cdot \zeta dx + \int_\Omega \text{grad } h \cdot \text{grad } \zeta dx = 0 .$$

Then choosing $\zeta = (u - h)^+$ and substracting the above expressions we obtain

$$\frac{1}{2}\frac{d}{dt}\|(u - h)^+\|_{L^2(\Omega)} + \int_\Omega \text{grad}(u - h) \cdot \text{grad}(u - h)^+ dx \leq 0 .$$

So $\|(u - h)^+(t)\|_{L^2(\Omega)} \leq \|(u - h)^+(0)\|_{L^2(\Omega)}$ holds which finishes the proof. ∎

Lemma 5. *Assume* $\psi \in L^2(\Omega)$, $\psi > 0$ *on* Ω *and* $u_{0,i} \in H_0^1(\Omega)$ $i = 1,2$. *Then if* u_i *is the weak solution of (1) corresponding to* $u_{0,i}$, *it follows that*

$$\|(u_1(t) - u_2(t))^+\|_{L^2(\Omega)} \leq \|(u_{0,1} - u_{0,2})^+\|_{L^2(\Omega)}$$

and

$$\|(u_1(t) - u_2(t))^-\|_{L^2(\Omega)} \leq \|(u_{0,1} - u_{0,2})^-\|_{L^2(\Omega)} .$$

In particular when $u_0 \geq 0$ (resp. $u_0 \leq 0$) a.e. on Ω we have $u(t) \geq 0$ (resp. $u(t) \leq 0$) a.e. on Ω and $t > 0$.

Proof. Immediate from the proof of the previous lemma taking $h(t,x,0,0)$ i.e. $h \equiv 0$. ∎

Remark 2. Better comparison results could be obtained using the fact that P is "well posed" on $L^\infty(\Omega)$ under supplementary hypotheses (on ψ and u_0).

The two following results are derived from the m-accretiveness in $H_0^1(\Omega)$ of the operator B as well as the theory of Variational Inequalities.

Proposition 3. *Let $\psi_i \in L^2(\Omega)$ with $\psi_i \geq 0$ a.e. on Ω for $i = 1,2$. Assume $u_0 \in H_0^1(\Omega)$ and let u^i be the weak solution of (1) corresponding to the obstacle ψ_i. Then $\psi_1 \leq \psi_2$ a.e. on Ω implies $u^1(t,x) \leq u^2(t,x)$ a.e. on $(0,\infty) \times \Omega$.*

Proof. Taking into account the definition of the $H_0^1(\Omega)$ semigroup solution it is enough to see that when $u^i(x) \in H_0^1(\Omega)$ verifies

$$u^i + \lambda B_{\psi_i} u^i = f$$

(with $f \in D(B_{\psi_i})$) then $u^1(x) \leq u^2(x)$ a.e. $x \in \Omega$. Proceeding as in Appendix 3 (Lemma A.8) we know that the functions $\tilde{u}_i = u^i - f$ are solutions of the Variational Inequality given by $\tilde{u}_i \in K_i = \{v \in H_0^1(\Omega) : v(x) \leq \lambda\psi_i(x)$ a.e. $x \in \Omega\}$ and

$$\int_\Omega \operatorname{grad} \tilde{u} \cdot \operatorname{grad}(v - \tilde{u})dx + \frac{1}{\lambda}\int_\Omega \tilde{u}(v - \tilde{u})dx \geq \langle \Delta f, v - u\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}$$

$\forall\, v \in K_i$. Therefore from Proposition 1.9 of Brezis [5] we get $\tilde{u}_1 \leq \tilde{u}_2$ a.e. on Ω and the proof ends. ∎

Proposition 4. *Let $\psi \in L^2(\Omega)$ with $\psi \geq 0$ a.e. on Ω. For $i = 1,2$, let $u_{0,i} \in H_0^1(\Omega)$ and denotes by u_i the associated weak solution of (1). Then $-\Delta u_{0,1} \leq 0 \leq -\Delta u_{0,2}$ in $D'(\Omega)$ implies $-\Delta u_1(t) \leq 0 \leq -\Delta u_2(t)$ in $D'(\Omega)$ a.e. $t > 0$.*

Proof. It is easy to see that $v^i = -\Delta u^i$, $i = 1,2$, are the $H^{-1}(\Omega)$ semigroup solutions of the Abstract Cauchy Problems

$$(20)\qquad \begin{cases} \dfrac{dv}{dt} + Ev \ni 0 \quad \text{in} \quad H^{-1}(\Omega), \quad \text{on} \quad (0,\infty) \\ v(0) = v_0 \end{cases}$$

corresponding to the initial data $v_0 = -\Delta u_{0,i}$, where E is the operator in $H^{-1}(\Omega)$ defined by

$$(21)\qquad E = (-\Delta)\circ B\circ(-\Delta)^{-1} .$$

(we recall that E is an m-accretive operator on $H^{-1}(\Omega)$, see Appendix 2). Then it is enough to prove that when $v_i(x) \in H^{-1}(\Omega)$ verify

$$v^i + \lambda E v^i = g_i \quad \text{on} \quad H^{-1}(\Omega) ,$$

($g_i \in H^{-1}(\Omega)$, being $g_1 \le g_2$ in $D'(\Omega)$ and g_1 or g_2 identicial to zero) then $v^1 \le v^2$ in $D'(\Omega)$. Arguing as in Appendix 2 it is easily seen that the function $h_i(x) = -\mathrm{Min}\{\psi(x),-u_i(x)\}$ is the solution of the Variational Inequality $h_i \in K^* = \{w \in H_0^1(\Omega) : w(x) \ge -\psi(x) \ \text{a.e.}\ x \in \Omega\}$ and

$$\lambda \int_\Omega \mathrm{grad}\, h_i \cdot \mathrm{grad}(w - h_i)dx + \int_\Omega h_i(w - h_i)dx \ge \langle g_i, w - h_i\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}$$

$\forall\, w \in K^*$. Therefore, by applying the Corollary I.5 of Brezis [5] we get $h^1 \le h^2$ a.e. on Ω. Finally the result follows from the fact that $g_j = 0$ implies $h_j = 0$. ∎

<u>Remark 3.</u> Better comparison results about $-\Delta u(t)$ could be obtained using the fact that P^* is well posed on $L^1(\Omega)$ under supplementary hypotheses (on ψ and u_0). The situation is similar to the one in Remark 2.

This section is finished with a curious and very useful estimate which is, out of slight modifications, a particular application of the abstract result of Benilan-Diaz [3].

<u>Proposition 4.</u> <u>Let</u> $\psi \in H^1(\Omega)$ <u>with</u> $\psi \ge 0$ <u>a.e. on</u> Ω <u>and</u> $(-\Delta\psi)^- \in L^2(\Omega)$. <u>Assume</u> $u_0 \in H_0^1(\Omega)$ <u>such that</u>

$$-\Delta u_0 \in \overline{D^+(A)}^{L^1(\Omega)} \qquad (D^+(A) = \{w \in D(A) : Aw \ge 0\}) .$$

<u>Then</u>

$$(23)\qquad h(t,x;0,\tilde{v}_0) \le -\mathrm{Min}\{\psi(x),\Delta u(t,x)\} \quad \underline{\text{a.e.}} \quad (t,x) \in (0,\infty)\times\Omega$$

<u>where</u> $\tilde{v}_0 = -\mathrm{Min}\{\psi,\Delta u_0\}$.

Proof. For $\alpha > 0$ let $a_0 \in D^+(A)$ be such that $\|-\Delta u_0 - a_0\|_{L^1(\Omega)} < \alpha$. Consider $a_1^n \in D(A)$ verifying

$$a_1^n + \lambda_n A a_1^n = a_0 \quad \text{for any} \quad n = 1,2,\dots .$$

By the T-accretiveness of the operator A we have

$$\|(a_1^n - a_0)^+\|_{L^1(\Omega)} \leq \|(a_1^n - a_0 + \lambda_n(\frac{a_0 - a_1^n}{\lambda_n} - z_0))^+\|_{L^1(\Omega)} = 0$$

if $z_0 \in Aa_0$. So $a_1^n \leq a_0$ and also $a_1^n \in D^+(A)$. Arguing by induction there exists $a_k^n \in D^+(A)$ such that $a_0 = a_0^n \geq a_1^n \geq a_2^n \geq \dots \geq a_k^n \geq \dots$ and

$$\frac{a_k^n - a_{k-1}^n}{\lambda_n} + Aa_k^n = 0 \qquad k = 1,2,\dots,n . \tag{24}$$

Set $w_k^n = \mathrm{Min}\{\psi, -a_k^n\}$. Then from (24) we have

$$w_k^n - \lambda_n \Delta w_k^n = f_k^n \tag{25}$$

where $f_k^n = a_{k-1}^n - a_k^n + w_k^n$. It is easy to check that

$$f_k^n = a_{k-1}^n + \mathrm{Min}\{\psi, -a_{k-1}^n\} - a_k^n - \mathrm{Min}\{\psi, -a_k^n\} - \mathrm{Min}\{\psi, -a_{k-1}^n\} \geq$$

$$\geq -\mathrm{Min}\{\psi, -a_{k-1}^n\} = w_{k-1}^n .$$

On the other hand, if we denote $h(t) = h(t,\cdot;0,\tilde{v}_0)$ then $h(t) = \lim_{n\to+\infty} h_n(t)$ with $h_n(t) = d_k^n$ if $\lambda_n k \leq t < \lambda_n(k+1)$, the elements $d_k^n \in \{w \in W_0^{1,1}(\Omega) : \Delta w \in L^1(\Omega)\}$ and satisfy

$$\begin{cases} \dfrac{d_k^n - d_{k-1}^n}{\lambda_n} - \Delta d_k^n = 0 \quad \text{in} \quad L^1(\Omega), \quad k = 1,2,\dots,n \\ d_0^n = \tilde{v}_0 \qquad \forall\, n \in \mathbb{N} . \end{cases} \tag{26}$$

Using the T-accretiveness of the operator $-\Delta$ on $L^1(\Omega)$ (i.e. the operator A given by (12) when $\beta(x,r) = r$) we deduce from (25) and (26)

$$(27) \qquad \|(d_k^n - w_k^n)^+\|_{L^1(\Omega)} \leq \|(d_{k-1}^n - f_k^n)^+\|_{L^1(\Omega)} \leq \ldots \leq \|(\tilde{v}_0 - w_0^n)^+\|_{L^1(\Omega)}$$

Now by the Crandall-Liggett Theorem ([11]) and the proof of Theorem 1 we know that if $v(t) = -\Delta u(t)$ then $v(t) = \lim_{n\to\infty} u_n(t)$ where $u_n(t) = a_k^n$ for $k\lambda_n \leq t < (k+1)\lambda_n$ and for T fixed we have the estimate

$$\max_{k=1,\ldots,k(n)} \max_{t\in[k\lambda_n,(k+1)\lambda_n)} \|v(t) - a_k^n\|_{L^1(\Omega)} \leq \alpha + (\lambda_n)^{1/2} T \cdot \|\Delta \operatorname{Min}\{\psi, a_0\}\|_{L^1(\Omega)}$$

where $k(n)$ is such that $|k(n)\lambda_n - T| < \lambda_n$. By the continuity in $L^1(\Omega)$ of the transformation $w \to -\operatorname{Min}\{\psi,-w\}$ we have

$$\max_{k=1,\ldots,k(n)} \max_{t\in[k\lambda_n,(k+1)\lambda_n)} \|-\operatorname{Min}\{\psi,-v(t)\} - w_k^n\|_{L^1(\Omega)} \leq \rho(n,\alpha)$$

with $\overline{\lim}_{\alpha\to\infty}\, \overline{\lim}_{n\to\infty} \rho(n,\alpha) = 0$. Then, we obtain (23) passing to the limit in (27) when $n \to \infty$ and $\alpha \to 0$. ∎

Remark 4. In Benilan-Diaz [13] it is proved that (23) is not true (in general) without the hypothesis (22).

§4. ON THE ASYMPTOTIC BEHAVIOUR.

Our attention is fixed, at the moment, on the convergence of the weak solution u to an equilibrium point of (1). It is clear that, in general, the asymptotic behaviour of u depends in an essential way of u_0 (for any fixed obstacle ψ). The following result improves that of Brezis in some particular cases:

Proposition 5. *Let $\psi \in L^2(\Omega)$ with $\psi \geq 0$ a.e. on Ω. Let $u_0 \in H_0^1(\Omega)$ and u be the weak solution of (1). The following holds:*

i) If $-\Delta u_0 \geq 0$ in $D'(\Omega)$ then $u(t) \to 0$ (strongly) in $H_0^1(\Omega)$ when $t \to \infty$ where u_∞ is a solution of (3).

ii) If $-\Delta u_0 \leq 0$ in $D'(\Omega)$ then $u(t) \to u_\infty$ (strongly) in $H_0^1(\Omega)$ when $t \to \infty$ where u_∞ is a solution of (3).

Proof: i) By Proposition 4 $-\Delta u(t) \geq 0$ in $D'(\Omega)$. Then $u(t)$ satisfies $u_t = \Delta u$ a.e. $t > 0$ and the conclusion holds from the results about the asymptotic behaviour for the linear heat equation. ii) In this case $-\Delta u(t) \leq 0$ in $D'(\Omega)$ by Proposition 4. Then it is easy to see that u is the solution of the problem

$$(28)\qquad \begin{aligned} &|u_t| \leq \psi && \text{on } (0,\infty) \times \Omega \\ &u_t - \Delta u = 0 && \text{on } \{(t,x) : |u_t(t,x)| < \psi(x)\} \\ &u_t - \Delta u \leq 0 && \text{on } \{(t,x) : u_t(t,x) = \psi(x)\} \\ &u_t - \Delta u \geq 0 && \text{on } \{(t,x) : u_t(t,x) = -\psi(x)\} \\ &u = 0 && \text{on } (0,\infty) \times \partial\Omega \\ &u(0,x) = u_0(x) && \text{on } \Omega \end{aligned}$$

and so it is well known that $u(t) \to u_\infty$ (strongly) in $H_0^1(\Omega)$ (see [8]). Finally, as $-\Delta u_\infty \leq 0$ in $D'(\Omega)$, u_∞ is a solution of (3). ∎

The next theorem is the main result of this Section and guarantees the strong convergence of the solution to zero.

__Theorem 3.__ __Assume__ $\psi \in H^2(\Omega)$ __with__ $\psi \geq 0$, $\Delta\psi \geq 0$ __a.e. on__ Ω __and let__ $u_0 \in H_0^1(\Omega)$. __Then if__ $\psi(x) > 0$ __a.e.__ $x \in \Omega$ $u(t) \to 0$ __(strongly) in__ $H_0^1(\Omega)$ __when__ $t \to \infty$. __If in addition__ $\psi(x) \geq \delta$ __for some__ $\delta > 0$ __then__ $u_t = \Delta u$ __on__ $(T_0,\infty) \times \Omega$ __where__ $T_0 = \left(\frac{C}{\delta} \|\psi\|_{L^1(\Omega)}\right)^{2/N}$ __and__ C_1 __a positive constant depending only on__ $|\Omega|$.

__Proof.__ __1^{st} step.__ Assume $\psi \in C^2(\overline{\Omega})$, $\psi \geq 0$, $\Delta\psi \geq 0$ and $u_0 \in H_0^1(\Omega)$ such that $h = -\Delta u_0 \in L^\infty(\Omega)$. Set $u_{0,+}$ and $u_{0,-}$ belonging to $H_0^1(\Omega)$ such that $-\Delta u_{0,+} = h^+$ and $-\Delta u_{0,-} = -h^-$. Let u_+ and u_- be the weak solutions of (1) corresponding to the initial data $u_{0,+}$ and $u_{0,-}$ respectively. By Theorem 2 and the T-accretiveness of A we know that

$$(29)\qquad -\Delta u_-(t) \leq -\Delta u(t) \leq -\Delta u_+(t) \quad \text{in } L^\infty(\Omega), \text{ a.e. } t > 0$$

From the Proposition 5 and the well known results on the asymptotic behaviour for the linear heat equation we deduce that $-\Delta u_+(t) \to 0$ in $L^\infty(\Omega)$ when $t \to +\infty$. On the other hand it is possible to find a $\hat{u}_0 \in H_0^1(\Omega)$ with $\Delta\hat{u}_0 \in L^\infty(\Omega)$ and such that $-\Delta\hat{u}_0 \leq -\Delta u_{0,-}$ a.e. on Ω as well as $-\Delta\hat{u}_0 \in \overline{D^+(A)}^{L^1(\Omega)}$. Indeed, it suffices to choose $\hat{v}_0 \in L^\infty(\Omega)$ such that $\hat{v}_0 \leq \mathrm{Min}\{-\psi,-\Delta u_{0,-}\}$ and then $\hat{u}_0 = (-\Delta)^{-1}\hat{v}_0$. (We remark that in this case $\mathrm{Min}\{\psi,\Delta\hat{u}_0\} = \psi$ so $A(-\Delta\hat{u}_0) = \Delta\psi \geq 0$). Therefore Proposition 4 shows that

$$(30)\qquad h(t,x;0,-\psi(x)) \leq -\mathrm{Min}\{\psi(x),\Delta\hat{u}(t,x)\} \leq -\mathrm{Min}\{\psi(x),\Delta u_-(t,x)\} \leq 0$$

where $\hat{u}$ is the weak solution of (1) corresponding to the initial datum $\hat{u}_0$. From the results on the asymptotic behaviour for the linear heat equation it is well known that

i) $\forall\, t > 0 \quad h(t,x;0,-\psi(x)) \in L^\infty(\Omega)$

ii) there exists a positive constant C (only depending on $|\Omega|$) such that

(31) $$\frac{-C}{t^{N/2}} \|\psi\|_{L^1(\Omega)} \leq h(t,x;0,-\psi(x)) \leq 0 \quad \text{a.e.} \quad (t,x) \in (0,\infty) \times \Omega$$

Estimates (30) and (31) shows that if $\psi(x) > 0$ a.e. $x \in \Omega$ then $\Delta u_-(t) \to 0$ in $L^\infty(\Omega)$ when $t \to +\infty$ and the first assertion follows from (29) and the fact that $-\Delta u(t) \to 0$ in $L^\infty(\Omega)$ implies $u(t) \to 0$ (strongly) in $H_0^1(\Omega)$ when $t \to \infty$. On the other hand, if $\psi(x) \geq \delta > 0$, from (30) and (31) it follows that

$$-\psi(x) \leq -\Delta u_-(t) \leq -\Delta u(t)$$

for $t \geq T_0$, $T_0 = [\frac{C}{\delta} \|\psi\|_{L^1(\Omega)}]^{2/N}$. Then $\mathrm{Min}\{\psi(x), \Delta u(t,x)\} = \Delta u(t,x)$ a.e. $(t,x) \in (T_0,\infty) \times \Omega$ and the second assertion holds.

<u>2^{nd} step.</u> Take $\psi \in C^2(\Omega)$ with $\psi \geq 0$ and $\Delta\psi \geq 0$ a.e. on Ω. Let $u_0 \in H_0^1(\Omega)$. Consider $u_{0,n} \in H_0^1(\Omega)$ with $-\Delta u_{0,n} \in L^\infty(\Omega)$ and $u_{0,n} \to u_0$ in $H_0^1(\Omega)$ when $n \to \infty$. Then if $u_n(t)$ is the weak solution of (1) of initial datum $u_{0,n}$ it is known that $u_n(t) \to u(t)$ in $H_0^1(\Omega)$ when $n \to \infty$ and so the first assertion follows from the first step. Besides when $\psi(x) \geq \delta > 0$, one has $(u_n)_t = \Delta u_n$ a.e. on $(T_0,\infty) \times \Omega$ with $T_0 = [\frac{C}{\delta} \|\psi\|_{L^1(\Omega)}]^{2/N}$. Therefore by the "exponential formule" (see e.g. [7], corolary 4.4) we have for $t \geq T_0$

$$u(t) = \lim_{m\to\infty} (I + \frac{t}{m} B)^{-m} u(T_0) = \lim_{m\to\infty} (I + \frac{t}{m} B)^{-m} (\lim_{n\to\infty} u_n(T_0)) =$$

$$= \lim_{n\to\infty} (\lim_{m\to\infty} (I + \frac{t}{m} B)^{-m} u_n(T_0)) = \lim_{n\to\infty} (\lim_{m\to\infty} (I + \frac{t}{m} (-\Delta))^{-m} u_n) T_0)) =$$

$$= \lim_{m\to\infty} (I + \frac{t}{m} (-\Delta))^{-m} u(T_0)$$

(we have identified $-\Delta$ with the m-accretive operator on $H_0^1(\Omega)$ of domain $\{z \in H_0^1(\Omega) : \Delta z \in H_0^1(\Omega)\}$). Therefore the second assertion holds.

<u>3^{rd} step.</u> Let $\psi \in H^2(\Omega)$ with $\psi \geq 0$, $\Delta\psi \geq 0$ a.e. on Ω and $u_0 \in H_0^1(\Omega)$. Consider $\psi_n \in C^2(\bar{\Omega})$ with $\Delta\psi_n \geq 0$ such that $\|\psi_n\|_{L^1(\Omega)} \leq \|\psi\|_{L^1(\Omega)}$ and $\psi_n \to \psi$ in $H^2(\Omega)$ when $n \to \infty$. Arguing as in the above step it is enough to prove that if u_n is the weak solution of (1) corresponding to the obstacle ψ_n then $u_n(t) \to u(t)$ (strongly) in $H_0^1(\Omega)$ when $n \to \infty$. By an abstract result of the theory of evolution equations (see [7] Theorem 4.2) it is sufficient to show that

$$(I + \lambda B_n)^{-1}z \to (I + \lambda B)^{-1}z \quad \text{when } n \to \infty, \ \forall \lambda > 0 \text{ and } \forall z \in D(B) \cap D(B_n) .$$

(B_n designates the operator B corresponding to the obstacle ψ_n). Setting $y_n = (I + \lambda B_n)^{-1}z$ and $y = (I + \lambda B)^{-1}z$ and arguing as in Appendix 2 we know that $\tilde{y}_n \equiv y_n - z$ satisfies $\tilde{y}_n \in K_n \equiv \{v \in H_0^1(\Omega) : v(x) \leq \lambda\psi_n(x)$ a.e. $x \in \Omega\}$ and

$$\int_\Omega \text{grad } \tilde{y}_n \cdot \text{grad}(v - \tilde{y}_n)dx + \frac{1}{\lambda}\int_\Omega \tilde{y}_n(v - \tilde{y}_n)dx \geq \langle \Delta z, v - u \rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}$$

$\forall v \in K_n$. Then by the results of the theory of Variational Inequalities (and thanks to the fact that $\psi_n \to \psi$ in $H^1(\Omega)$) we obtain $\tilde{y}_n \to \tilde{y} = y - z$ (strongly) in $H_0^1(\Omega)$ when $n \to \infty$. ■

<u>Remark 5.</u> The above result improves a previous one of [15] concerning the case $\psi(x) \equiv \delta > 0$ a.e. $x \in \Omega$.

When $\psi(x) \geq \delta > 0$ but without any additional regularity hypotheses we don't know if the identification, after a finite time, between u and a solution of the linear heat equation occurs or not. Nevertheless the following result shows that in this case the asymptotic behaviour is not very different.

<u>Proposition 6. Let $\psi \in L^2(\Omega)$ with $\psi(x) \geq \delta$ a.e. on Ω, for some $\delta > 0$, and $u_0 \in H_0^1(\Omega)$. Then, with the notation of the Lemma 4, we have</u>

(32) $\quad h(t,x;T_0,u_\delta(T_0,x)) \leq u(t,x) \leq h(t,x;0,u_0(x))$ <u>a.e.</u> $(t,x) \in (T_0,\infty) \times \Omega$,

<u>where u_δ is the solution of (1) corresponding to $\psi(x) \equiv \delta$ and T_0 is given in Theorem 3. In particular</u> $u(t) \to 0$ <u>(strongly) in</u> $L^p(\Omega)$ <u>for every</u> $1 \leq p \leq +\infty$ <u>when</u> $t \to +\infty$.

Proof. From Theorem 3 $u_\delta(t,\cdot) = h(t,\cdot;T_0,u_\delta(T_0,\cdot))$ with $T_0 = (C\cdot|\Omega|)^{2/N}$. Then Proposition 3 and Lemma 4 lead to the estimate (32). So $u(t) \to 0$ (strongly) in $L^p(\Omega)$ for every $1 \leq p \leq +\infty$. ∎

We consider now (in some particular cases) the problem of choosing $\lim_{t\to\infty} u(t)$ among all the equilibrium points of (1).

Proposition 7. *Let $\psi \in L^2(\Omega)$ with $\psi \geq 0$ a.e. on Ω, and $u_0 \in H_0^1(\Omega)$. Then the following holds:*

a) *if $u_0 \geq 0$ a.e. on Ω, $u(t,x) = 0$ a.e. $x \in \{x \in \Omega : u_0(x) = 0$ and $\psi(x) = 0\}$, $\forall\, t > 0$. Moreover $\lim_{t\to\infty} u(t,x) = 0$ (weakly) in $H_0^1(\Omega)$*

b) *if $\Delta u_0 \leq 0$ in $D'(\Omega)$, $u(t,x) = u_0(x)$ a.e. $x \in \{x \in \Omega : \psi(x) = 0\}$ $\forall\, t > 0$.*

Proof. a) By Lemmas 4 and 5 it follows

$$0 \leq u(t,x) \leq u_0(x) + t\cdot\psi(x) .$$

On the other hand

$$0 \leq u(t,x) \leq h(t,x;0,u_0)$$

which implies that $u(t,x) \to 0$ (strongly) in $L^2(\Omega)$ when $t \to \infty$. Then if u_∞ is the weak limit point of $u(t)$ when $t \to \infty$ due to the compactness of the inclusion $H_0^1(\Omega) \subset L^2(\Omega)$ we deduce that $u(t,x) \to u_\infty$ (strongly) in $L^2(\Omega)$. Part b) is a consequence of the fact that $u(t,x) \geq u_0(x)$ as it can be checked from the definition of u. Then the conclusion holds by the Lemma 4. ∎

Part a) of the previous result shows that if the measure of the set $\{x \in \Omega : \psi(x) = 0\}$ is positive then the second assertion of Theorem 3 is not possible. Part b) gives a simple situation where $\lim_{t\to\infty} u(t)$ is not identically zero.

Remark 6. The equation of problem (1) can obviously be written as

$$u_t + \mathrm{Max}\{-\Delta u, -\psi\} = 0$$

and then it is similar to the so called Bellman's equation of Dynamic Programming (see e.g. [1]). It would be interesting to know if our results can be proved (or improved) by stochastic arguments.

Remark 7. In [8] the study of the asymptotic behaviour of the solutions of the problem (28) is also proposed. Our methods remain still valid and its application is left to the reader.

APPENDIX 1. Basic Theory of Accretive Operators

Given a Banach space X and an operator $A : D(A) \subseteq X \to P(X)$ we call $u \in C([0,\infty) : X)$ a semigroup solution of the Abstract Equation

(A.1) $$\frac{du}{dt} + Au \ni 0 \quad \text{on} \quad (0,\infty)$$

if there exists $\lambda_n > 0$, $\lambda_n \to 0$ when $n \to \infty$ and a sequence $\{a_k^n\}$ $k = 0,1,\ldots$ satisfying

(A.2) $$\frac{a_k^n - a_{k-1}^n}{\lambda_n} + Aa_k^n \ni 0, \qquad k = 1,2,\ldots,n \in \mathbb{N}$$

and such that the sequence $u_n(t)$ defined by $u_n(t) = a_k^n$ if $k\lambda_n \leq t < (k+1)\lambda_n$ verify $\|u(t) - u_n(t)\| \leq \lambda_n$.

Roughly (A.2) represents a simple implicit Euler approximation of (A.1) and we are defining solutions of (A.1) to be limits of solutions of these difference approximations.

There are many criteria ensuring the existence of the λ_n-approximate solution u_n, being a simple one the following "range condition":

(A.3) $$R(I + \lambda A) \supseteq D(A) \quad \forall \lambda > 0$$

(see details in the survey article of Crandall [10]). The question of the convergence of such a sequence lead to the notion of accretive operator.

Definition A.1. An operator $A : D(A) \subseteq X \to P(X)$ is called accretive if $\forall [x,y],[\hat{x},\hat{y}] \in A$

(A.4) $$\|(x - \hat{x})\| \leq \|x - \hat{x} + \lambda(y - \hat{y})\| \quad \text{for all} \quad \lambda > 0 \,.$$

If X is also a Banach lattice then A is called T-accretive if $\forall [x,y],[\hat{x},\hat{y}] \in A$

(A.5) $$\|(x - \hat{x})^+\| \leq \|(x - \hat{x} + \lambda(y - \hat{y}))^+\| \quad \text{for all } \lambda > 0 \,.$$

where $h^+ = \max(h,0)$.(*) Finally if A satisfies (A.4) and $R(I + \lambda A) = X$ $\forall \lambda > 0$ A is called m-accretive.

Proposition A.1. Let A be accretive (resp. T-accretive). Let $u_0 \in \overline{D(A)}$. If there exists an λ_n-approximate solution u_n of (A.1) such that $\|u_n(0) - u_0\| \leq \lambda_n$ then there

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(*) If X is a normal Banach lattice (i.e. $\|u^+\| \leq \|v^+\|$ and $\|u^-\| \leq \|v^-\|$ implies $\|u\| \leq \|v\|$) then any T-accretive operator is also accretive.

exists $u \in C([0,\infty) : X)$ semigroup solution of (A.1) such that $u(0) = u_0$. Moreover if u and $\hat{u}$ are semigroup solutions of (A.1) then

$$\|u(t) - \hat{u}(t)\| \leq \|u(0) - \hat{u}(0)\| \quad (\text{resp. } \|(u(t) - \hat{u}(t))^+\| \leq \|(u(0) - \hat{u}(0))^+\|)$$

i.e. the application $S(t)u_0 = u(t)$ is a semigroup of contractions on $\overline{D(A)}$.

This proposition is proved in [12] (resp. in [2]) for accretive (resp. T-accretive) operators. In both works more sophisticated situations are also considered.

Let us introduce some notation that provides an alternative characterization of accretiveness and T-accretiveness (often easier to verify in practice than (A.4) and (A.5)). For $x, y \in X$, define

$$\tau(x,y) = \inf_{\lambda>0} \frac{\|x + \lambda y\| - \|x\|}{\lambda}$$

$$\sigma(x,y) = \sup_{\lambda<0} \frac{\|x + \lambda y\| - \|x\|}{\lambda}$$

and also

$$\tau^+(x,y) = \inf_{\lambda>0} \frac{\|(x + \lambda y)^+\| - \|x^+\|}{\lambda}, \quad \sigma^+(x,y) = \sup_{\lambda<0} \frac{\|(x + \lambda y)^+\| - \|x^+\|}{\lambda}$$

when X is assumed to be a Banach lattice. It is easy to check that A is accreive (resp. T-accretive) if and only if

$$\tau(x - \hat{x}, y - \hat{y}) \geq 0 \quad (\text{resp. } \tau^+(x - \hat{x}, y - \hat{y}) \geq 0)$$

for all $[x,y],[x,y] \in A$. If A satisfies the stronger assumption

$$\sigma(x - \hat{x}, y - \hat{y}) \geq 0 \quad (\text{resp. } \sigma^+(x - \hat{x}, y - \hat{y}) \geq 0)$$

for all $[x,y],[\hat{x},\hat{y}] \in A$, A is called strongly accretive (resp. strongly T-accretive). It is well-known (see [10]) that a densely defined, linear and accretive (resp. T-accretive) operator in a strongly accretive (resp. T-accretive) one.

The advantage of these alternative characterization is that for certain spaces X the above products are easy to compute:

__Lemma A.1.__ (c.f. Sato [19]). __Let__ $\Omega \subset \mathbb{R}^N$

i) __if__ $X = L^p(\Omega)$ __for__ $1 < p < \infty$

$$\tau^+(f,g) = \sigma^+(f,g) = \begin{cases} \dfrac{1}{\|f^+\|^{p-1}} \displaystyle\int_\Omega |f|^{p-1} \cdot \mathrm{sign}_0^+ f \cdot g \, dx & f \not\equiv 0 \\ \|g^+\| & f \equiv 0 \end{cases}$$

ii) __if__ $X = L^1(\Omega)$

$$\tau^+(f,g) = \max\{\int \alpha \cdot g \cdot dx,\ \alpha \in L^\infty,\ \alpha(x) \in \mathrm{sign}^+ f(x) \ \text{ a.e.}\}$$

$$\sigma^+(f,g) = \min\{\int \alpha \cdot g \cdot dx,\ \alpha \in L^\infty,\ \alpha(x) \in \mathrm{sign}^+ f(x) \ \text{ a.e.}\}$$

iii) __if__ $X = L^\infty(\Omega)$

$$\tau^+(f,g) = \max\{\lim_{\lambda \downarrow 0} \text{ess sup}\big[\alpha(x) g(x) : x \in \Omega(f,\lambda)\big], \alpha \in L^\infty, \alpha(x) \in \mathrm{Sign}^+ f(x) \text{ a.e.}\}$$

$$\sigma^+(f,g) = \min\{\lim_{\lambda \downarrow 0} \text{ess inf}\big[\alpha(x) g(x) : x \in \Omega(f,\lambda)\big], \alpha \in L^\infty, \alpha(x) \in \mathrm{sign}^+ f(x) \text{ a.e.}\}$$

__where__ $\Omega(f,\lambda) = \{x \in \Omega : |f(x)| > \|f\|_{L^\infty} - \lambda\}$ __and__

$$\mathrm{sign}_0^+(v) = \begin{cases} 1 & \text{if } v > 0 \\ 0 & \text{if } v \leq 0 \end{cases}, \qquad \mathrm{sign}^+(v) = \begin{cases} 1 & \text{if } v > 0 \\ [0,1] & \text{if } v = 0 \\ 0 & \text{if } v < 0 . \end{cases} \quad \blacksquare$$

When X is a Hilbert space of scalar product $(\ ,\ )$ it is easy to see that A is accretive if and only if A is __monotone__ (i.e. $(x - \hat{x}, y - \hat{y}) \geq 0 \quad \forall\ [x,y],[\hat{x},\hat{y}] \in A$). In this case the classes of m-accretive operators and of maximal monotone ones coincide (see [7]).

APPENDIX 2. <u>The Problem P^* (β given by (5)) is well posed on $L^1(\Omega)$ and $H^{-1}(\Omega)$.</u>

The main aim of this appendix is to prove that P^* is a well posed problem on $L^1(\Omega)$ (when $\psi \in H^1(\Omega)$, $\psi \geq 0$, $(-\Delta\psi)^- \in L^2(\Omega)$ and β is given by (5)); that is the statement of Proposition 1.

<u>Lemma A.2.</u> <u>Assume</u> $\psi \in L^2(\Omega)$, $\psi \geq 0$ <u>a.e. on</u> Ω <u>and consider the operator</u> A <u>on</u> $L^1(\Omega)$ <u>given by (12) i.e.</u>

$$D(A) = \{w \in L^1(\Omega) : \beta(x,w(x)) \in W_0^{1,1}(\Omega) \text{ and } \Delta\beta(x,w(x)) \in L^1(\Omega)\}$$

$$Aw = -\Delta\beta(\cdot,w(\cdot)) \text{ if } w \in D(A) .$$

<u>Then</u> A <u>is T-accretive in</u> $L^1(\Omega)$.

<u>Proof.</u> The operator $-\Delta$ defined in $L^1(\Omega)$ by $D(-\Delta) = \{w \in W_0^{1,1}(\Omega) : \Delta w \in L^1(\Omega)\}$ is a strongly T-accretive operator in $L^1(\Omega)$. Then for any $u^* \in D(-\Delta)$ and any $\alpha(x) \in L^\infty(\Omega)$ such that $\alpha(x) \in \operatorname{sign}^+ u^*(x)$ a.e. $x \in \Omega$, we have

$$\text{(A.6)} \qquad \int_\Omega -\Delta u^* \cdot \alpha dx \geq 0 .$$

Now let $[u,v],[\hat{u},\hat{v}] \in A$ (i.e. $u,\hat{u} \in D(A)$ and $v = -\Delta\beta(x,u)$, $\hat{v} = -\Delta\beta(x,\hat{u})$). Then $u^* = \beta(\cdot,u) - \beta(\cdot,\hat{u})$ belongs to $D(-\Delta)$. Taking

$$\alpha^*(x) = \begin{cases} 1 & \text{if } (u-\hat{u})(x) \geq 0 \text{ and } u^*(x) \geq 0 \\ 0 & \text{if } (u-\hat{u})(x) \leq 0 \text{ and } u^*(x) < 0 \text{ or } (u-\hat{u})(x) < 0 \text{ and } u^*(x) = 0 \end{cases}$$

then $\alpha^*(x) \in L^\infty(\Omega)$ and $\alpha^*(x) \in \operatorname{sign}^+(u(x) - \hat{u}(x)) \cap \operatorname{sign}^+ u^*(x)$. So

$$\int_\Omega (Au - A\hat{u})\alpha^* dx \geq 0$$

by (A.6), which shows the T-accretiveness of operator A. ∎

Our next step is to prove the range condition (A.5) which is only well known for ψ <u>not depending on</u> x (see Brezis-Strauss [9]). For it we begin with a technical lemma

<u>Lemma A.3.</u> <u>Assume</u> ψ <u>to be a measurable function on</u> Ω <u>and let</u> β <u>given by (5). Then</u>

$$\text{(A.7)} \qquad \beta^{-1}(x,r) = r + \gamma(r + \psi(x)) \quad \forall r \in D(\beta^{-1}(x,\cdot)), \text{ a.e. } x \in \Omega$$

<u>being</u> $\gamma(r)$ <u>the maximal monotone graph of</u> R^2 <u>defined by</u>

(A.8) $\gamma(r) = 0$ *if* $r > 0$, $\gamma(0) = (-\infty,0]$ *and* $\gamma(r) = \phi$ *(the empty set) if* $r < 0$.

Proof. If $r > -\psi(x)$ it is clear that $\beta^{-1}(x,r) = r + \gamma(r + \psi(x))$. If $a \in \beta^{-1}(x,r)$ with $r = -\psi(x)$ then $a = r + (a - r)$ where $(a - r) \geq 0$ and so $(a - r) \in \gamma(0)$. Conversely if $a = r + h$ $(h \in \gamma(0))$ then $-\mathrm{Min}\{\psi(x),-x\} = r = -\psi(x)$ and $a \in \beta^{-1}(x,r)$. ∎

Lemma A.4. *Let* $\psi \in H^1(\Omega)$ *be such that* $\psi \geq 0$ *a.e. on* Ω *and* $\Delta\psi$ *is a measure with* $(-\Delta\psi)^- \in L^2(\Omega)$. *Let* β *given by (5). Then the operator* A *is m-accretive in* $L^1(\Omega)$. *More concretely, for all* $f \in L^1(\Omega)$ *there exists a unique* $u \in L^1(\Omega)$ *with* $\beta(x,u) \in W_0^{1,q}(\Omega)$ $\left(1 \leq q < \frac{N}{N-1}\right)$ *such that*

$$(\text{A.9}) \qquad \begin{cases} u(x) - \lambda\Delta\beta(x,u(x)) = f(x) & \text{a.e. on } \Omega \\ \beta(x,u(x)) = 0 & \text{on } \partial\Omega . \end{cases}$$

Proof. Set $h(x) = \beta(x,u(x))$. Then u is a solution of (A.9) if and only if $h \in W_0^{1,1}(\Omega)$, $\Delta h \in L^1(\Omega)$ and $-\lambda\Delta h(x) + \beta^{-1}(x,h(x)) \ni f(x)$ a.e. $x \in \Omega$, or equivalently (by Lemma A.3)

$$(\text{A.10}) \qquad \begin{cases} -\lambda\Delta h(x) + h(x) + \gamma(h(x) + \psi(x)) \ni f(x) & \text{a.e. } x \in \Omega \\ h = 0 & \text{on } \partial\Omega . \end{cases}$$

Due to the accretiveness of operator $(-\lambda\Delta + I)$ on $L^1(\Omega)$ and from the monotonicity of γ we know that if $\hat{h} \in D(-\Delta)$ is the solution of (A.10) corresponding to $\hat{f} \in L^1(\Omega)$ then

$$(\text{A.11}) \qquad \|[f - (-\lambda\Delta h + h)] - [\hat{f} - (-\lambda\Delta\hat{h} + \hat{h}]\|_{L^1(\Omega)} \leq \|f - \hat{f}\|_{L^1(\Omega)}$$

(see Brezis-Strauss [9]). In particular, the coercivity of the operator $(-\lambda\Delta + I)$ in $L^1(\Omega)$ implies that

$$(\text{A.12}) \qquad \alpha\|h - \hat{h}\|_{L^1(\Omega)} \leq \|-\lambda\Delta(h - \hat{h}) + (h - \hat{h})\|_{L^1(\Omega)} \leq 2\|f - \hat{f}\|_{L^1(\Omega)} \quad \text{for some } \alpha > 0 .$$

From (A.12) the uniqueness follows. To prove the existence of solution it suffices to consider f being in a dense set of $L^1(\Omega)$. Indeed, let h_n (with $-\lambda\Delta h_n + h_n \in L^1(\Omega)$) be the solution of (A.10) corresponding to f_n and $f_n \to f$ in $L^1(\Omega)$. By (A.12) we have

$$\alpha\|h_n - h_m\|_{L^1(\Omega)} \leq \|-\lambda\Delta(h_n - h_m) + (h_n - h_m)\|_{L^1(\Omega)} \leq 2\|f_n - f_m\|_{L^1(\Omega)}$$

and then $h_n \to h$ and $-\lambda\Delta h_n + h_n \to -\lambda\Delta h + h$; finally $f - (-\lambda\Delta h + h) \in \gamma(u + \psi)$ since γ is maximal.

Actually, when $f \in L^2(\Omega)$ and $h \in H_0^1(\Omega)$ is the solution of the Variational Inequality

$$(A.13)\qquad \begin{cases} h(x) \geq -\psi(x) \quad \text{a.e.} \quad x \in \Omega \\ -\lambda\Delta h + h \geq f \quad \text{a.e. on} \quad \Omega \\ (h + \psi)(-\lambda\Delta h + h - f) = 0 \quad \text{a.e. on} \quad \Omega \\ h = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

it is well known (due to the hypothesis on ψ) that $h \in H^2(\Omega) \cap H_0^1(\Omega)$ (see Brezis [5]). Therefore h satisfies (A.10). Finally the function $u = f + \lambda\Delta h - \lambda$ is such that $h(x) \in \beta(x,u(x))$ a.e. $x \in \Omega$ and so u is the solution of (A.9). (The regularity on $\beta(x,u(x))$ come from the fact that $\Delta\beta(\cdot,u(\cdot)) \in L^1(\Omega)$, see [9]). ∎

<u>Lemma A.5 Assume</u> ψ <u>and</u> β <u>as in Lemma A.4. Then</u> $\overline{D(A)}^{L^1(\Omega)} = L^1(\Omega)$.

<u>Proof.</u> It is enough to see $L^\infty(\Omega) \subset \overline{D(A)}^{L^1(\Omega)}$. Take $f \in L^\infty(\Omega)$ and for each $\lambda > 0$ let $z_\lambda \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of (A.10). By Theorem I.1 of Brezis [5] we get

$$(A.14)\qquad \|\lambda\Delta z_\lambda\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} + C\cdot\|(-\lambda\Delta\psi)^-\|_{L^2(\Omega)}$$

with C independent of λ. Therefore $\{\lambda z_\lambda\}$ converges weakly in $H^2(\Omega)$ and then strongly in $L^2(\Omega)$, when $\lambda \to 0$. But $\{\lambda z_\lambda\} \to 0$ in $L^2(\Omega)$ because $\|z_\lambda^+\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)}$ (by the comparison results) and then $\|z_\lambda\|_{L^2(\Omega)} \leq C'$, C' independent of λ (because $-\psi(x) \leq -z_\lambda^-(x) \leq z_\lambda(x)$ a.e. $x \in \Omega$). Setting $y_\lambda(x) = f(x) + \lambda\Delta z_\lambda(x)$ it is clear the $y_\lambda(x) \in \beta^{-1}(x,z_\lambda(x))$ a.e. $x \in \Omega$ (see Lemma A.3), $y_\lambda \in D(A)$ and y_λ converges (weakly) to f in $L^2(\Omega)$ when $\lambda \to 0$. Finally from (A.14) we deduce that $\lim_{\lambda\to 0} \|y_\lambda\|_{L^2(\Omega)} = \|f\|_{L^2(\Omega)}$ and then y_λ converges (strongly) in $L^2(\Omega)$. ∎

The proof of Proposition 1 is now a consequence of Proposition A1 and Lemmas A2, A4 and A5. Problem P* is also well posed on the space $H^{-1}(\Omega)$:

<u>Lemma A.6. Assume</u> $\psi \in L^2(\Omega)$, $\psi \geq 0$ <u>a.e. on</u> Ω. <u>Consider the operator</u> E <u>on</u> $H^{-1}(\Omega)$ <u>given by (21), i.e.</u>

$$E = (-\Delta)\circ B\circ(-\Delta)^{-1} .$$

<u>Then</u> E <u>is m-accretive in</u> $H^{-1}(\Omega)$.

<u>Proof.</u> Recalling that the scalar product in $H^{-1}(\Omega) = (H_0^1(\Omega))'$ is given by

$$(f,g)_{H^{-1}\times H^{-1}} = ((-\Delta)^{-1}f,(-\Delta)^{-1}g)_{H_0^1\times H_0^1}$$

then if $[x,y],[\hat{x},\hat{y}] \in E$

$$(x - \hat{x}, y - \hat{y})_{H^{-1}\times H^{-1}} = ((-\Delta)^{-1}(x - \hat{x}),(-\Delta)^{-1}(y - \hat{y}))_{H_0^1\times H_0^1} \geq 0$$

because $[(-\Delta)^{-1}x,(-\Delta)^{-1}y]$, $[(-\Delta)^{-1}\hat{x},(-\Delta)^{-1}\hat{y}] \in B$. Analogously $R(I + \lambda B) = H_0^1(\Omega)$ implies $R(I + \lambda E) = H^{-1}(\Omega)$, $\forall \lambda > 0$. ∎

The following unpublished result of A. Damlamian characterizes the operator E when $\psi \in H_0^1(\Omega)$.

<u>Proposition A.2.</u> <u>Assume</u> $\psi \in H_0^1(\Omega)$ <u>with</u> $\psi \geq 0$ <u>a.e. on</u> Ω. <u>Then</u> $E = \partial\Phi$, <u>where</u> Φ <u>is the convex, l.s.c. function defined in</u> $H^{-1}(\Omega)$ by

$$\Phi(u) = \begin{cases} -\frac{1}{2}\|\psi\|^2_{L^2(\Omega)} + \langle u,-\psi\rangle_{H^{-1}\times H_0^1} + \frac{1}{2}\|(u + \psi)^+\|^2_{L^2(\Omega)} & \underline{\text{if}}\quad u \in L^2(\Omega)^+ - H^{-1}(\Omega)^- \\ +\infty & \underline{\text{otherwise}} . \end{cases}$$

<u>Remark A.1.</u> When $\psi(x) \equiv \delta$, the function $\tilde{v} = v + \delta$ (v solution of P*) coincides with the solution of the one-phase Stefan Problem

$$\begin{cases} \tilde{v}_t - \Delta\theta(\tilde{v}) = 0 & \text{on } (0,\infty)\times\Omega \\ \tilde{v} = \delta & \text{on } (0,\infty)\times\partial\Omega \\ \tilde{v}(0,\cdot) = v_0(\cdot) + \delta & \text{on } \Omega \end{cases}$$

where $\theta(r) = 0$ if $r \leq -\delta$ and $\theta(r) = r + \delta$ if $r \geq -\delta$. In this case, formulation (1) coincides with the one given in [20] (see also the Appendix of [13]).

APPENDIX 3. The Problem P is Well Posed on $L^\infty(\Omega)$.

The accretiveness in $L^\infty(\Omega)$ of the operator C given in (17), i.e.,

$$D(C) = \{w \in L^\infty(\Omega) \cap H_0^1(\Omega) : \Delta w \in L^\infty(\Omega),\ \mathrm{Min}\{\psi,\Delta w\} \in L^\infty(\Omega)\}$$

$$Cw = -\mathrm{Min}\{\psi,\Delta w\} \quad \text{if} \quad w \in D(c) \ ,$$

is an application of the abstract result of [4] or [17]. Here we show it directly.

Lemma A.7. Assumed $\psi \in L^2(\Omega)$, $\psi > 0$ a.e. on Ω, the operator C is T-accretive in $L^\infty(\Omega)$.

Proof. Let $[u,v],[\hat{u},\hat{v}] \in C$. Let us assume that $w = (u - \hat{u})^+ \not\equiv 0$. Then if β is given by (5) $\tau^+(w,\beta(x,-\Delta u) - \beta(x,-\Delta\hat{u})) > 0$ because otherwise for some $\lambda > 0$ one would have $\beta(x,-\Delta u) - \beta(x,-\Delta\hat{u}) < 0$ a.e. on $\Omega(w,\lambda)$. From the monotonicity of $\beta(x,\cdot)$ we would deduce that $-\Delta(u - \hat{u}) \leq 0$ a.e. on $\Omega(w,\lambda)$. But $(u - \hat{u}) = \|w\|_{L^\infty} - \lambda$ in the boundary of $\Omega(w,\lambda)$ and the application of the maximum principle would lead to a contradiction. ∎

About the range condition one has:

Lemma A.8. Assumed $\psi \in C^2(\overline{\Omega})$ with $\psi > 0$ on $\overline{\Omega}$, C satisfies the range condition (A.3). More exactly, for all $f \in L^\infty(\Omega) \cap H_0^1(\Omega)$ such that $\Delta f \in L^\infty(\Omega)$ there exists $u \in D(C)$ solution of

$$u + \lambda Cu = f \quad \text{if} \quad \lambda > 0 \ . \tag{A.15}$$

Moreover

$$\|\Delta u\|_{L^\infty(\Omega)} \leq c(\|\Delta f\|_{L^\infty(\Omega)} + \|\Delta\psi\|_{L^\infty(\Omega)}) \tag{A.16}$$

for some constant c independent of λ and f.

Proof. Set $\tilde{u} = u - f$. Then it is easy to see that u is a solution of (A.15) if and only if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\Delta\tilde{u} \in L^\infty(\Omega)$ and $\tilde{u}$ satisfies $-\Delta\tilde{u}(x) - \beta^{-1}\left(x,-\frac{\tilde{u}(x)}{\lambda}\right) \ni \Delta f(x)$ a.e. $x \in \Omega$, or equivalently (see Lemma A.3)

$$\begin{cases} -\Delta\tilde{u}(x) + \dfrac{\tilde{u}(x)}{\lambda} - \gamma\left(\psi(x) - \dfrac{\tilde{u}(x)}{\lambda}\right) \ni \Delta f(x) & \text{a.e.} \quad x \in \Omega \\ \tilde{u} = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{A.17}$$

where γ is the graph given by (A.8). Problem (A.17) coincides with the Variational Inequality

$$\text{(A.18)}\qquad \begin{cases} \tilde{u}(x) \leq \lambda\psi(x) & \text{a.e. } x \in \Omega \\ -\Delta\tilde{u} + \dfrac{\tilde{u}}{\lambda} \leq \Delta f & \text{a.e. on } \Omega \\ (\tilde{u} - \lambda\psi)(-\Delta\tilde{u} + \dfrac{\tilde{u}}{\lambda} - \Delta f) = 0 & \text{a.e. on } \Omega \\ \tilde{u} = 0 & \text{on } \partial\Omega . \end{cases}$$

It is well known that under the assumption $\psi \in C^2(\overline{\Omega})$, $\psi > 0$ on $\overline{\Omega}$ there exists a unique $\tilde{u} \in H_0^1(\Omega)$ solution of (A.18) satisfying $\Delta\tilde{u} \in L^\infty(\Omega)$ and $\|\lambda\Delta\tilde{u}\|_{L^\infty(\Omega)} \leq C(\|\lambda\Delta f\|_{L^\infty(\Omega)} + \|\lambda\Delta\psi\|_{L^\infty(\Omega)})$ (see [18]). Then $u \in D(C)$ and it verifies (A.15) and (A.16). ■

Problem P can actually be "solved" in terms of the Proposition 1. Indeed, when $u_0 \in D(C)$, by Proposition A.1 and Lemmas A.7 and A.8 there exists a unique $u \in L^\infty(\Omega)$ semigroup solution of P. Finally the regularity of u follows from Theorem 2 of Benilan-Ha [4] (Notice that the hypothesis $R(\beta(x,\cdot)) = R$ a.e. $x \in \Omega$ made in [4] is only used to prove the existence of the $L^\infty(\Omega)$ semigroup solution).

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This paper deals with the first boundary problem associated with the fully nonlinear equation $u_t = \mathrm{Min}\{\psi, \Delta u\}$ on the set $\Omega \times (0,\infty)$, where Ω is a domain of $\mathbb{R}^N$ and $\psi(x)$ is a given obstacle such that $\psi \geq 0$ on Ω. Formulating the problem (occurring in heat control) as an Evolution Variational Inequality, H. Brezis obtained the existence and uniqueness of

(continued)

ABSTRACT (continued)

weak solutions in the space $H_0^1(\Omega)$ as well as weak convergence to an unknown equilibrium point of the equation (when t goes to infinity). The strong convergence of the solution to the zero equilibrium point is shown here, provided the obstacle is positive and subharmonic. If in addition $\psi(x) \geq \delta > 0$ then the asymptotic behaviour is completely described in the sense that the solution satisfies the *linear* heat equation $u_t = \Delta u$ on $(T_0,\infty) \times \Omega$, T_0 being a finite time. To do this the results are first presented for *strong* solutions (that is, those which satisfy the equation a.e.). The fact that under more regularity on the initial datum the weak solution is also a strong one and certain useful comparison principles are proved by using the theory of accretive operators in Banach spaces.